



# The Gabriel–Roiter measure and representation types of quivers

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## Abstract

A GR segment of an Artin algebra is a sequence of Gabriel–Roiter measures that is closed under direct successors and direct predecessors. The number of GR segments was conjectured to relate to the representation types of finite-dimensional hereditary algebras. We prove in the paper that a path algebra  $KQ$  of a finite connected acyclic quiver  $Q$  over an algebraically closed field  $K$  is of wild representation type if and only if  $KQ$  admits infinitely many GR segments.

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## 1. Introduction

Throughout, by Artin algebras or finite-dimensional algebras we always mean connected ones, that is, 0 and the identity are the only central idempotents. We first recall what Gabriel–Roiter measures are [14,15]. Let  $\mathbb{N} = \{1, 2, \dots\}$  be the set of natural numbers and  $\mathcal{P}(\mathbb{N})$  be the set of all subsets of  $\mathbb{N}$ . A total order on  $\mathcal{P}(\mathbb{N})$  can be defined as follows: if  $I, J$  are two different subsets of  $\mathbb{N}$ , write  $I < J$  if the smallest element in  $(I \setminus J) \cup (J \setminus I)$  belongs to  $J$ . Let  $A$  be an Artin algebra. A finitely generated left  $A$ -module is simply called a module. For a  $A$ -module  $M$ , we denote its length by  $|M|$ . The *Gabriel–Roiter (GR) measure* of  $M$  is defined

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to be the maximum (in  $\mathcal{P}(\mathbb{N})$ ) of the sets  $\{|M_1|, |M_2|, \dots, |M_t|\}$ , where  $M_1 \subset M_2 \subset \dots \subset M_t$  is a chain of indecomposable submodules of  $M$ . If  $M$  itself is an indecomposable  $\Lambda$ -module, then an indecomposable submodule  $X$  of  $M$  is called a *GR submodule* in case  $\mu(M) = \mu(X) \cup \{|M|\}$ , thus if and only if every proper submodule of  $M$  has GR measure at most  $\mu(X)$ . The corresponding factor  $M/X$  is always indecomposable [14].

A subset  $I$  of  $\mathbb{N}$  is called a GR measure of  $\Lambda$  if there is an indecomposable  $\Lambda$ -module  $M$  with GR measure  $\mu(M) = I$ . (From now on,  $\mu$  also denotes a GR measure when a module is not specified.) Using the GR measure, Ringel obtained a partition of the module category for any Artin algebra of infinite representation type [14]: there are infinitely many GR measures  $I_i$  and  $I^i$  with  $i$  natural numbers, such that

$$I_1 < I_2 < I_3 < \dots \quad \dots < I^3 < I^2 < I^1$$

and such that any other GR measure  $I$  satisfies  $I_i < I < I^i$  for all  $i$ . The GR measures  $I_i$  (resp.  $I^i$ ) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module is called a take-off (resp. central, landing) module if its GR measure is a take-off (resp. central, landing) measure.

Let  $\mu, \mu'$  be two GR measures of  $\Lambda$ . We call  $\mu'$  a *direct successor* of  $\mu$  (or  $\mu$  a *direct predecessor* of  $\mu'$ ) if, first,  $\mu < \mu'$  and second, there does not exist a GR measure  $\mu''$  with  $\mu < \mu'' < \mu'$ . The so-called *Successor Lemma* in [15] states that any GR measure  $\mu$  different from  $I^1$ , the maximal one, has a direct successor. However, there may exist GR measures (not the minimal one  $I_1$ ), which do not admit direct predecessors. The Hasse diagram of the GR measures of  $\Lambda$  consists of the following data: the vertices are the GR measures, and there is an edge between  $\mu$  and  $\mu'$  if  $\mu'$  is a direct successor of  $\mu$ .

**Definition 1.1.** A sequence of GR measures of an Artin algebra  $\Lambda$  is called a GR segment if it is closed under direct successors and direct predecessors, i.e. the Hasse subdiagram containing all of these GR measures is a connected component.

The following characterization for representation-finite Artin algebras is a direct consequence of Ringel's partition theorem and the Successor Lemma.

**Lemma 1.2.** Let  $\Lambda$  be an Artin algebra. Then the following are equivalent:

- (1)  $\Lambda$  is of finite representation type.
- (2)  $\Lambda$  admits only one GR segment.
- (3)  $\Lambda$  admits a finite GR segment.

Therefore, each GR segment of a representation-infinite Artin algebra is an infinite sequence. An infinite GR segment is indexed by natural numbers  $\mathbb{N}$ ,  $-\mathbb{N}$  or integers  $\mathbb{Z}$  in an obvious way. A GR segment  $\mathcal{S}$  is  $\mathbb{N}$ -indexed if and only if there is a GR measure  $\mu \in \mathcal{S}$  such that  $\mu$  does not admit a direct predecessor. By the Successor Lemma, a GR segment is  $-\mathbb{N}$ -indexed if and only if it consists of the landing measures.

Let  $K$  be an algebraically closed field. By a quiver we mean a finite connected acyclic quiver  $Q$ . We denote by  $KQ$  the (finite-dimensional) path  $K$ -algebra of  $Q$ . We recall from [9] that a quiver is of finite representation type if and only if its underlying undirected graph is a Dynkin graph (of type  $A_n, D_n, E_6, E_7$  or  $E_8$ ). We recall from [6,11] that a quiver  $Q$  is of tame representation type if and only if its underlying undirected graph is an extended Dynkin graph (of type  $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$  or  $\tilde{E}_8$ ). A quiver, which is neither finite nor tame type, is of wild representation type. (We refer to [5,7] for the notions of *finite*, *tame*, and *wild* for finite-dimensional algebras.)

It was shown for a tame quiver  $Q$  that the GR measures, not being the minimal one  $I_1$ , admitting no direct predecessors exist and that there are only finitely many GR segments ([2], see also Section 2.2). An affirmative answer of the following conjecture will obviously give an alternative characterization of tameness and wildness for quivers.

**Conjecture 1.3.** *Let  $K$  be an algebraically closed field and  $\Lambda = KQ$  a finite-dimensional path  $K$ -algebra for a connected acyclic quiver  $Q$ . Then the following are equivalent:*

- (1)  $\Lambda$  is of wild representation type.
- (2)  $\Lambda$  admits infinitely many GR segments.
- (3) There are infinitely many GR measures having no direct predecessors.

A similar problem has been considered by Ringel and Fahr [8]. They considered modules of finite length as well as modules of infinite length. Ringel conjectured that there are only countably many GR measures in the case of a tame algebra. There exists an unpublished proof for the tame hereditary case. In [8] Fahr proved that for a 3-Kronecker quiver there are uncountably many infinite GR measures. This trivially implies the existence of infinitely many GR segments. In [4] it was proved for the wild quiver  $1 \longrightarrow 2 \rightrightarrows 3$  that there are infinitely many GR measures which do not admit direct predecessors.

In this paper, we are going to construct, using the same method in [4], infinitely many GR measures without direct predecessors for wild  $n$ -Kronecker quivers and to verify the first two statements of Conjecture 1.3.

**Theorem 1.4.** *Let  $K$  be an algebraically closed field. The finite-dimensional path  $K$ -algebra  $KQ$  of a finite connected acyclic quiver  $Q$  is of wild representation type if and only if it admits infinitely many GR segments.*

## 2. Proof of the theorem

Before giving a proof of Theorem 1.4, some known results will be recalled. We refer to [1,13] as general references of representation theory of finite-dimensional algebras. Throughout, let  $K$  denote an algebraically closed field.

### 2.1. Gabriel's Main Property

The so-called *Gabriel's Main Property* on GR measure was proved in [14].

**Proposition 2.1.** *Let  $\Lambda$  be an Artin algebra and  $X$  and  $Y_1, \dots, Y_m$  be indecomposable  $\Lambda$ -modules. Assume that  $f : X \rightarrow \bigoplus Y_i$  is a monomorphism.*

- (1)  $\mu(X) \leq \max\{\mu(Y_i)\}$ .
- (2) If  $\mu(X) = \max\{\mu(Y_i)\}$ , then  $f$  splits.

### 2.2. Tame quivers

Let  $\Lambda = KQ$  be a path algebra of a tame quiver  $Q$ . Let  $X$  be a quasi-simple module of rank  $r$ , that is,  $r$  is the minimal natural number such that  $\tau^r X \cong X$ . If  $r = 1$ ,  $X$  is called a homogeneous simple module and denoted by  $H$ . Notice that there are infinitely many isomorphism classes of homogeneous simple modules and they all have the same GR measure. Let  $b$  be the number of isomorphism classes of the quasi-simple modules  $X$  with rank  $r > 1$  and  $a$  be the number of those such that  $\mu(X[r]) \geq \mu(H)$ . The following proposition summarizes the main results in [2].

**Proposition 2.2.** *Let  $\Lambda = KQ$  be a path algebra of a tame quiver  $Q$ .*

- (1) *The GR measure of the homogeneous simple modules lies in an  $\mathbb{N}$ -indexed GR segment.*
- (2) *The number of GR segments is bounded by  $b + 3$ .*
- (3) *The number of  $\mathbb{Z}$ -indexed GR segments is bounded by  $a$ .*

### 2.3. Proof of the theorem

To complete the proof of [Theorem 1.4](#) it is sufficient, by [Proposition 2.2](#), to show the existence of infinitely many GR segments for wild quivers.

**Theorem 2.3.** *Let  $Q$  be a finite connected acyclic wild quiver. Then  $KQ$  has infinitely many GR segments.*

**Proof.** We show that for any indecomposable regular  $KQ$ -module  $X$ , there is an indecomposable regular  $KQ$ -module  $Y$  with  $\mu(X) < \mu(Y)$  such that  $\mu(X)$  and  $\mu(Y)$  are not in the same GR segment. We refer to [10, Sections 4 and 10] for some known results on the homomorphisms between regular modules, in particular, Proposition 10.7 and Lemma 4.6 in there.

It is well-known and easy to see that  $KQ$  has a module  $N$  with  $\text{End}(N) = K$  and  $\text{Ext}^1(N, N) \neq 0$  (for example, one may consider a tame subquiver  $Q'$  of  $Q$  and take for  $N$  a homogeneous simple  $KQ'$ -module). For a given  $X$ , there is some  $n \geq 0$  such that  $\text{Hom}(X, \tau^n N)$  contains a monomorphism (here  $\tau$  denotes the Auslander–Reiten translation). Note that  $M = \tau^n N$  is again a module with  $\text{End}(M) = K$  and  $\text{Ext}^1(M, M) \neq 0$ . For this  $M$  there is some  $m \geq 0$  such that  $Y = \tau^m M$  has the following properties:

- (1)  $M$  is cogenerated by  $Y$ ,
- (2)  $\text{Ext}^1(M, Y) = 0$ .

Let  $\mathcal{F}(M)$  be the category of modules having a filtration with all factors isomorphic to  $M$ . The process of simplification [12] asserts that this category  $\mathcal{F}(M)$  is a hereditary length category of infinite representation type with a unique simple object, namely  $M$ . Since  $\text{Ext}^1(M, M) \neq 0$ , there is a chain of proper inclusion maps of indecomposable modules  $M_i$ , say

$$M = M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_t \rightarrow \cdots$$

where all  $M_i$  belong to  $\mathcal{F}(M)$ . On the other hand, if  $M'$  belongs to  $\mathcal{F}(M)$ , then it can be shown using induction and properties (1) and (2) that  $M'$  is cogenerated by  $Y$ . But this implies (Gabriel's Main Property) that the GR measures of  $M$  (in particular,  $X$ ) and  $Y$  belong to different GR segments. This completes the proof.

The induction procedure completes the proof of [Theorem 1.4](#).  $\square$

**Remark.** The proof of [Theorem 2.3](#) presented here is actually given by an anonymous referee. The author's original proof consists of two parts. For wild quivers with at least three vertices, the existence of regular tilting modules plays an essential role. Actually for  $X$  and a regular tilting module  $T$ , one may find some  $r \geq 0$  such that  $\text{Hom}(\tau^r T, \tau^{-m} X) = 0$  for all  $m \geq 0$ . It follows that all indecomposable modules  $X[i]$  sitting on the ray starting with  $X$  are cogenerated by  $T_1 = \tau^{r+1} T$ . Thus  $\mu(X)$  and  $\mu(T_1)$  (equals to the maximal GR measure of the direct summands of  $T_1$ ) belong to different GR segments. For wild quivers with two vertices, infinitely many GR measures without direct predecessors, equivalently, infinitely many  $\mathbb{N}$ -indexed GR segments, are directly constructed (see Section 3).

### 3. Wild $n$ -Kronecker quivers

The so-called  $n$ -Kronecker quivers  $Q$  are  $2 \begin{array}{c} \xrightarrow{\alpha_1} \\ \vdots \\ \xrightarrow{\alpha_n} \end{array} 1$  with  $n \geq 1$  arrows. An

indecomposable  $KQ$  module is simply called an  $n$ -Kronecker module. Note that  $Q$  is of finite representation type if  $n = 1$  and of tame type if  $n = 2$ . If  $n \geq 3$ , then  $Q$  is of wild representation type. In this section, we are going to use the same method in [4] to construct infinitely many GR measures admitting no direct predecessors for  $n$ -kronecker quivers with  $n \geq 3$ .

**Example.** As an example, we describe the GR measures of 2-Kronecker quiver  $\tilde{\mathbb{A}}_1$ . They are used in our later discussion.

- (1) The take-off modules are precisely the indecomposable preprojective modules as well as the simple injective module. The take-off measures are  $\{1, 3, 5, \dots, 2m + 1\}$  ( $m \geq 0$ ).
- (2) An indecomposable module is a central module if and only if it is regular. The central measures are  $\{1, 2, 4, 6, \dots, 2m\}$  ( $m \geq 1$ ) and  $\mu(H_1) = \{1, 2\}$  does not admit a direct predecessor.
- (3) The landing modules are precisely the non-simple indecomposable preinjective modules. The landing measures are  $\{1, 2, 4, \dots, 2m, 2m + 1\}$  ( $m \geq 1$ ).

**Remark.** The above partition is actually true for all  $n$ -Kronecker quivers. A detailed description can be found in [3, Proposition 2.3]. The take-off part of a bimodule algebra  $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$  was described in [16].

**Lemma 3.1.** *Let  $Q$  be a wild  $n$ -Kronecker quiver and  $X$  an indecomposable  $KQ$ -module.*

- (1) *The GR measure  $\mu(X) = \{1, 2\}$  if and only if  $\underline{\dim} X = (1, 1)$ .*
- (2) *The GR measure  $\mu(X) = \{1, 2, 3\}$  if and only if  $\underline{\dim} X = (2, 1)$ .*
- (3) *If the dimension vector  $\underline{\dim} X = (1, 1)$ . Then for each  $i \geq 1$  the GR measure  $\mu(\tau^i X)$  starts with  $\{1, 2, 3\}$ , i.e.,  $\mu(\tau^i X) = \{1, 2, 3, \dots\}$ .*

**Proof.** Statements (1) and (2) are straightforward. (3) First notice that an indecomposable regular module with dimension vector  $(1, 1)$  or  $(2, 1)$  has no proper regular factor modules. Using the Coxeter matrix and the Euler Form (see, for example, [13, Chapter 2]) one obtains  $\text{Hom}(X, \tau X) \neq 0$ . Therefore, there are monomorphisms from  $X$  to  $\tau X$ , thus monomorphisms  $\tau^i X \rightarrow \tau^{i+1} X$  for all  $i \geq 0$ . In particular,  $\mu(\tau^i X) > \mu(\tau^j X)$  for  $i > j \geq 0$ . On the other hand, it is not difficult to show that  $\tau X$  contains some indecomposable module with dimension vector  $(2, 1)$ . Thus  $\mu(\tau^i X) \geq \mu(\tau X) > \{1, 2, 3\}$  for all  $i \geq 1$ .  $\square$

**Lemma 3.2.** *Let  $Q$  be a wild  $n$ -Kronecker quiver and  $M$  an indecomposable  $KQ$ -module. Let  $\mu_m = \{1, 2, 4, \dots, 2m\}$  and  $\mu^m = \{1, 2, 4, \dots, 2m, 2m + 1\}$  for  $m \geq 1$ .*

- (1) *For each  $m \geq 1$ ,  $\mu_m$  and  $\mu^m$  are (central) GR measures.*
- (2) *If  $\mu(M) = \mu_m$ , then  $\underline{\dim} M = (m, m)$ .*
- (3) *If  $\mu(M) = \mu^m$ , then  $\underline{\dim} M = (m + 1, m)$ .*
- (4) *If  $\mu(M) = \mu_m$  or  $\mu^m$ , then any indecomposable regular factor of  $M$  contains some indecomposable submodule with dimension vector  $(1, 1)$ .*

**Proof.** Since 2-Kronecker modules are obvious  $n$ -Kronecker modules, both  $\mu_m$  and  $\mu^m$  are GR measures. By induction and the fact that a GR factor of length 2 has dimension vector  $(1, 1)$ , it

is easily seen that  $\underline{\dim} M = (m, m)$  if  $\mu(M) = \mu_m$ . Similarly,  $\underline{\dim} M = (m + 1, m)$  holds if  $\mu(M) = \mu^m$ .

(4) This is clear for  $m = 1$ . Let  $m \geq 2$  and  $M \xrightarrow{\pi} Y$  be a proper epimorphism with regular module  $Y$  regular and kernel  $K$ . Let  $N$  be a GR submodule of  $M$  with inclusion map  $f$ :

$$\begin{array}{ccccccc} & & & N & & & \\ & & & \downarrow f & & & \\ 0 & \longrightarrow & K & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y \longrightarrow 0 \end{array}$$

We first assume that  $\mu(M) = \mu_m$ . If the composition  $\pi f$  is not zero, then the image of  $\pi f$  is a regular factor of  $N$ . Therefore by induction,  $\text{im } \pi f$ , and thus  $Y$ , contains a submodule with dimension vector  $(1, 1)$ . If the composition  $\pi f$  is zero, then  $f$  factors through  $\iota$ . In particular, there is a monomorphism  $N \xrightarrow{g} K$ . Since  $N$  is a GR submodule of  $M$ , it follows that  $N$  is isomorphic to a direct summand of  $K$  (Proposition 2.1). Since  $Y$  is not simple and  $|K| \leq |N| = 2m - 2$ , one has  $K \cong N$ . Thus  $|Y| = 2$  and  $\underline{\dim} Y = (1, 1)$ .

If  $\mu(M) = \mu^m$ , then  $\mu(N) = \mu_m$ . Since the factor  $M/N$  is the injective simple module, the composition  $N \xrightarrow{\pi f} Y$  is not zero. Therefore, the image of  $\pi f$ , and thus  $Y$ , contains a submodule with dimension vector  $(1, 1)$ .  $\square$

**Theorem 3.3.** *Let  $Q$  be a wild  $n$ -Kronecker quiver. Then for each  $m \geq 1$ , the GR measure  $\mu^m = \{1, 2, 4, \dots, 2m, 2m + 1\}$  does not admit a direct predecessor. Therefore, there are infinitely many  $\mathbb{N}$ -indexed GR segments.*

**Proof.** For the purpose of a contradiction, we assume that  $M$  is an indecomposable module such that  $\mu(M)$  is a direct predecessor of  $\mu^m$  for a fixed  $m \geq 1$ . Then  $M$  is a regular module since  $\mu^m$ , and thus  $\mu(M)$ , is a central measure. Assume  $M \cong X[r]$  for some quasi-simple module  $X$  and  $r \geq 1$ . Since  $\mu_t < \mu^m$  for all  $t \geq 1$ ,  $\mu_m < \mu(M) < \mu^m = \mu_m \cup \{2m + 1\}$ . It follows that  $|X[r + 1]| > |X[r]| = |M| > 2m + 1$ . In particular  $\mu(M) = \mu(X[r]) < \mu^m < \mu(X[r + 1])$ . It is easily seen by the definition of GR measure that  $\mu(X[r + 1])$  has the form  $\{1, 2, \dots, 2t, 2t + 1, \dots\}$  for some  $1 \leq t \leq m$ . In particular,  $X[r + 1]$  contains an indecomposable submodule  $Y$  with GR measure  $\mu^t$  for some  $1 \leq t \leq m$ . Note that  $\underline{\dim} Y = (t + 1, t)$  and  $\mu(Y) \geq \mu^m$ .

We claim that  $\text{Hom}(Y, \tau^{-r} X) = 0$ . If this were not the case, then by Lemma 3.2(4), the image of a nonzero homomorphism, in particular  $\tau^{-r} X$ , would contain a submodule  $Z$  with dimension vector  $(1, 1)$ . Therefore, there would be a monomorphism  $\tau^r Z \rightarrow X$ , and thus, by Lemma 3.1,  $\mu^m > \mu(M) = \mu(X[r]) \geq \mu(X) > \mu(\tau^r Z) > \{1, 2, 3\}$ .

Since  $\text{Hom}(Y, \tau^{-r} X) = 0$  and the factor of the inclusion  $X[r] \subset X[r + 1]$  is isomorphic to  $\tau^{-r} X$ , the inclusion  $Y \rightarrow X[r + 1]$  factors through  $X[r]$ . Therefore, there is a monomorphism  $Y \rightarrow X[r]$ . It follows that  $\mu(X[r]) \geq \mu(Y) = \mu^t \geq \mu^m > \mu(M) = \mu(X[r])$ . This contradiction implies that  $\mu^m$  does not admit a direct predecessor for any  $m \geq 1$ .  $\square$

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